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GOODNESS–OF–FIT TESTS FOR MULTIVARIATE LAPLACE DISTRIBUTIONS

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Abstract. Consistent goodness-of-fit tests are proposed for multivariate Laplace distributions of arbitrary dimension. The test statistics are formulated following the Fourier–type approach of measuring the discrepancy between the empirical and the theoretical characteristic function, and result in computationally convenient representations. In the symmetric case interesting limit values are obtained which are related to well known measures of multivariate skewness and kurtosis. A Monte Carlo study is conducted in order to compare the new procedures with standard tests based on the empirical distribution function. A real data application is also included.

Keywords. Multivariate Laplace distribution; Goodness–of–fit test; Empirical characteristic function, Multivariate skewness; Monte Carlo test.

AMS 2000 classification numbers: 62H15, 62H12

1 Introduction

Testing goodness–of–fit of a given set of observations to a specific probabilistic model is a crucial aspect of data analysis. This problem has received wide attention in the univariate setting, but when handling multivariate data it appears that there is a lack of methods for distributions other than the normal. On the other hand, researchers in various fields have questioned the appropriateness of the Gaussian distribution in modeling certain empirical data which often exhibit skewness and tails heavier than those expected under the Gaussian assumption. An alternative probabilist model which has been popular lately is the multivariate Laplace distribution of Kotz et al. (2001). This model, and extensions thereof, was found appropriate for financial, biological and engineering data; see Kotz et al. (2001), Kozubowski and Podgórski (2001), Lindsey and Lindsey (2006), Eltoft et al. (2006), and Rossi and Spazzini (2010).

For univariate data there are standard goodness–of–fit tests which utilize the empirical distribution function (EDF). Due to the lack of proper ordering of vectors however, it is well known that these tests can not be generalized to handle multivariate observations in a straightdforward manner; refer to Justel et al. (1997) and Chiu and Liu (2009). The Gaussian case has nevertheless received considerable attention and several methods for goodness–of–fit testing have been devised that are particularly tailored for multinormality; see for instance, Kankainen et al. (2007), Liang and Ng (2009), Villasenor Alva and Estrada (2009), and Sürücü (2009).

In this paper we extend to the multivariate case the approach in Meintanis (2004) and propose a goodness–of–fit test for the composite null hypothesis of a multivariate Laplace distribution (MLD). We shall consider the symmetric case first. To fix notation let $\mathbf{X}$ be a random vector of dimension $d \geq 1$, and write $\phi(\mathbf{t}) := \mathbb{E}(e^{i\mathbf{t}'\mathbf{X}})$, $\mathbf{t} \in \mathbb{R}^d$, for the characteristic function (CF) of $\mathbf{X}$. The CF of the MLD is given by

$$(1.1) \qquad \varphi(\mathbf{t}) := \varphi(\mathbf{t}; \boldsymbol{\delta}, \boldsymbol{\Sigma}) = \frac{e^{i\mathbf{t}'\boldsymbol{\delta}}}{1 + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}},$$

where $\boldsymbol{\delta} \in \mathbb{R}^d$ and $\boldsymbol{\Sigma} \in \mathbb{M}_d$ denotes a scale matrix which belongs to the set $\mathbb{M}_d$ of positive definite matrices of order $d \times d$. We shall write $\mathbf{X} \sim \mathrm{ML}(\boldsymbol{\delta}, \boldsymbol{\Sigma})$ when $\mathbf{X}$ follows a MLD with CF given by (1.1). On the basis of independent observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$

on $\mathbf{X}$ we wish to test the null hypothesis

$$H_0\colon\ \mathbf{X} \sim \mathrm{ML}(\boldsymbol{\delta}, \boldsymbol{\Sigma}),\ \text{for some } \boldsymbol{\delta} \in \mathbb{R}^d \text{ and some } \boldsymbol{\Sigma} \in \mathbb{M}_d,$$

against general alternatives. Since the family of MLD is closed under full rank affine transformations we shall consider test statistics which involve the standardized data

$$\mathbf{Z}_j := \widehat{\boldsymbol{\Sigma}}_n^{-1/2}(\mathbf{X}_j - \widehat{\boldsymbol{\delta}}_n),\ \ j = 1, \ldots, n,$$

where $(\widehat{\boldsymbol{\delta}}_n, \widehat{\boldsymbol{\Sigma}}_n)$ denote consistent estimators of $(\boldsymbol{\delta}, \boldsymbol{\Sigma})$. Under the null hypothesis H_0, the data $\mathbf{Z}_j,\ j = 1, \ldots, n$, are for large n approximately distributed as $\mathrm{ML}(\mathbf{0}_d, \mathbf{I}_d)$ with $\mathbf{0}_d$ and $\mathbf{I}_d$ denoting the zero vector, and the identity matrix, respectively.

In view of (1.1) we suggest the test statistic

$$(1.2)\qquad T_{n,W} = n \int_{\mathbb{R}^d} \left| \varphi_n(\mathbf{t}) \left(1 + \frac{1}{2}\mathbf{t}'\mathbf{t}\right) - 1 \right|^2 W(\mathbf{t}) d\mathbf{t},$$

where $\varphi_n(\mathbf{t}) = n^{-1} \sum_{j=1}^n e^{i\mathbf{t}'\mathbf{Z}_j}$ is the empirical CF computed from the standardized observations $\mathbf{Z}_j,\ j = 1, \ldots, n$, and $W(\cdot)$ denotes a weight functions which is introduced in order to smooth out the periodic parts of $\varphi_n(\mathbf{t})$.

The rest of the paper is organized as follows. In Section 2 we discuss some computational aspects of the test statistics while Section 3 deals with the problem of estimation of parameters and the consistency of the test, while in Section 4 limit statistics are obtained and their moment connections are discussed. The finite–sample behavior of the proposed method compared to other procedures is studied by means of Monte Carlo in Section 5. Finally Section 6 is devoted to extension of the test statistic to the asymmetric case and to application with real data.

2 Computation of test statistic

In order to compute the test statistic, first write eqn. (1.2) as

$$(2.1)\qquad T_{n,W} = n \int_{\mathbb{R}^d} |D_n(\mathbf{t})|^2\ W(\mathbf{t}) d\mathbf{t},$$

where

$$D_n(\mathbf{t}) = \varphi_n(\mathbf{t}) \left(1 + \frac{1}{2}\|t\|^2\right) - 1,$$

and notice that

$$|D_n(\mathbf{t})|^2 = |\varphi_n(\mathbf{t})|^2 \left(1 + \frac{1}{2}\|t\|^2\right)^2 + 1 - 2\left(1 + \frac{1}{2}\|t\|^2\right) C_n(\mathbf{t}),$$

where $|\varphi_n(\mathbf{t})|^2 = C_n^2(\mathbf{t}) + S_n^2(\mathbf{t})$ and $C_n(\cdot)$ and $S_n(\cdot)$ denote the real and the imaginary part of the empirical CF, respectively. Upon writing

$$C_n(\mathbf{t}) = \frac{1}{n}\sum_{j=1}^{n} \cos(\mathbf{t}'\mathbf{Z}_j) \quad \text{and} \quad |\varphi_n(\mathbf{t})|^2 = \frac{1}{n^2}\sum_{j,k=1}^{n} \cos(\mathbf{t}'(\mathbf{Z}_j - \mathbf{Z}_k))$$

and by some extra algebra the test statistic is rendered in the convenient form

$$\text{(2.2)} \quad T_{n,W} = \frac{1}{n}\sum_{j,k=1}^{n} \left\{ I_1(\mathbf{Z}_{jk}) + I_2(\mathbf{Z}_{jk}) + \frac{1}{4} I_3(\mathbf{Z}_{jk}) \right\}$$

$$+ nI_1(\mathbf{0}_d) - \sum_{j=1}^{n} \left\{ 2I_1(\mathbf{Z}_j) + I_2(\mathbf{Z}_j) \right\},$$

where $\mathbf{Z}_{jk} = \mathbf{Z}_j - \mathbf{Z}_k$ and

$$I_m(\mathbf{x}) = \int_{\mathbb{R}^d} (\mathbf{t}'\mathbf{t})^{m-1} \cos(\mathbf{t}'\mathbf{x}) W(\mathbf{t}) d\mathbf{t}, \; m = 1, 2, 3.$$

A simple closed formula for the statistic in (2.2) may be obtained if we set $W(\mathbf{t}) = e^{-a\|\mathbf{t}\|^2}$, $a > 0$, by using the well known integral

$$\text{(2.3)} \quad \int_{\mathbb{R}^d} \cos(\mathbf{t}'\mathbf{x}) e^{-a\|\mathbf{t}\|^2} d\mathbf{t} = \left(\frac{\pi}{a}\right)^{d/2} e^{-\|\mathbf{x}\|^2/4a} := N_a(\mathbf{x}).$$

In particular, successive differentiation of (2.3) with respect to a, yields after some tedious but straightforward algebra,

$$\int_{\mathbb{R}^d} (\mathbf{t}'\mathbf{t}) \cos(\mathbf{t}'\mathbf{x}) e^{-a\|\mathbf{t}\|^2} d\mathbf{t} = \frac{1}{4a^2} \Delta_a(\mathbf{x}) N_a(\mathbf{x})$$

and

$$\int_{\mathbb{R}^d} (\mathbf{t}'\mathbf{t})^2 \cos(\mathbf{t}'\mathbf{x}) e^{-a\|\mathbf{t}\|^2} d\mathbf{t} = \frac{1}{16a^4} \left(8a\Delta_a(\mathbf{x}) + \Delta_a^2(\mathbf{x}) - 8a^2 d\right) N_a(\mathbf{x}),$$

where $\Delta_a(\mathbf{x}) = (2ad - \|\mathbf{x}\|^2)$. Then with weight function $e^{-a\|\mathbf{t}\|^2}$ the test statistic in (2.2), say $T_{n,a}$, admits the following representation

$$\text{(2.4)} \quad T_{n,a} = nN_a(\mathbf{0}_d) - \frac{1}{4a^2}\sum_{j=1}^{n} N_a(\mathbf{Z}_j) \left\{ \Delta_a(\mathbf{Z}_j) + 8a^2 \right\}$$

$$+\frac{1}{64a^4n}\sum_{j,k=1}^{n} N_a(\mathbf{Z}_{jk})\left\{\left[\Delta_a(\mathbf{Z}_{jk})+4a+8a^2\right]^2-8a^2(8a+d+2)\right\},$$

which is suitable for computer implementation. Notice also that since $T_{n,a}$ depends solely on $\|\cdot\|^2$, not even the computation of the square root of $\widehat{\mathbf{\Sigma}}_n^{-1}$ is required in (2.4). Other weight functions are also possible. In particular assume that $W(\cdot)$ may be decomposed into a product as $W(\mathbf{t}) = \prod_{m=1}^{d} w(t_m)$ (t_m, $m = 1, ..., d$, being the elements of $\mathbf{t}$), where the function $w(\cdot)$ satisfies $w(t) = w(-t)$, $t \in \mathbb{R}$, and $\int t^4 w(t)dt < \infty$. Then the test statistic in (1.2) can be written in closed form provided that the integral $\int t^m w(t)dt$, $m = 0, 2, 4$, is explicit. Such a weight function results for instance by letting $w(t) = e^{-a|t|}$, $a > 0$, but then the desirable feature of $T_{n,W}$ being dependent solely on $\|\cdot\|^2$ is lost. For more details see Meintanis and Iliopoulos (2008).

3 Estimation of parameters and consistency

The test statistic in (1.2) involves the standardized data $\mathbf{Z}_j$, $j = 1, \ldots, n$, which in turn depend on estimates $(\widehat{\boldsymbol{\delta}}_n, \widehat{\mathbf{\Sigma}}_n)$ of the parameters $(\boldsymbol{\delta}, \mathbf{\Sigma})$. We shall use the classical moment estimates. Specifically if $\mathbf{X} \sim \text{ML}(\boldsymbol{\delta}, \mathbf{\Sigma})$, it may be shown that

$$\mathbb{E}(\mathbf{X}) = \boldsymbol{\delta}, \quad \text{and} \quad \mathbb{E}\left[(\mathbf{X}-\boldsymbol{\delta})(\mathbf{X}-\boldsymbol{\delta})'\right] = \mathbf{\Sigma}.$$

Hence the moment estimates are given by the simple sample mean and the sample covariance matrix, respectively,

$$(3.1) \qquad \widehat{\boldsymbol{\delta}}_n = \overline{\mathbf{X}}_n \quad \text{and} \quad \widehat{\mathbf{\Sigma}}_n = \mathbf{S}_n,$$

where $\mathbf{S}_n = n^{-1}\sum_{j=1}^{n}(\mathbf{X}_j - \overline{\mathbf{X}}_n)(\mathbf{X}_j - \overline{\mathbf{X}}_n)'$. These estimates enjoy the properties

$$\widehat{\boldsymbol{\delta}}_n(\mathbf{Y}_1, \ldots, \mathbf{Y}_n) = \mathbf{A}\widehat{\boldsymbol{\delta}}_n(\mathbf{X}_1, \ldots, \mathbf{X}_n) + \boldsymbol{\beta}$$

and

$$\widehat{\mathbf{\Sigma}}_n(\mathbf{Y}_1, \ldots, \mathbf{Y}_n) = \mathbf{A}\widehat{\mathbf{\Sigma}}_n(\mathbf{X}_1, \ldots, \mathbf{X}_n)\mathbf{A}',$$

where $\boldsymbol{\beta} \in \mathbb{R}^d$ and $\mathbf{A}$ denotes a nonsingular matrix of order $d \times d$. This feature combined with the fact that in eqn. (2.4) only $\|\cdot\|^2$ appears, renders the test statistic affine invariant, i.e., $T_{n,a}$ satisfies

$$T_{n,a}(\mathbf{Y}_1, \ldots, \mathbf{Y}_n) = T_{n,a}(\mathbf{X}_1, \ldots, \mathbf{X}_n),$$

for each transformation of the type $\mathbf{X}_j \mapsto \mathbf{Y}_j = \mathbf{A}\mathbf{X}_j + \boldsymbol{\beta}, \; j = 1, ..., n$. Due to this property of the test statistic, in the simulations only the standard case $\mathrm{ML}(\mathbf{0}_d, \mathbf{I}_d)$ needs to be considered.

We now turn to the consistency of the test in eqn. (1.2) implement by using the moment estimators.

Theorem 3.1 *Let* $\mathbf{X}$ *be a random vector with mean* $\mathbb{E}(\mathbf{X}) = \boldsymbol{\mu}$ *which possesses a non–singular covariance matrix* $\mathbf{C}$*, and assume that the weight function satisfies*

$$\int_{\mathbb{R}^d} \|\mathbf{t}\|^4 W(\mathbf{t}) d\mathbf{t} < \infty.$$

Then as $n \to \infty$*,*

$$\text{(3.2)} \quad \frac{T_{n,W}}{n} \to \int_{\mathbb{R}^d} \left| e^{-i\mathbf{t}'\mathbf{C}^{-1/2}\boldsymbol{\mu}} \, \phi\left(\mathbf{C}^{-1/2}\mathbf{t}\right) \left(1 + \frac{1}{2}\|t\|^2\right) - 1 \right|^2 W(\mathbf{t}) d\mathbf{t} := \Delta_W,$$

almost surely.

Proof. From (2.1) we have

$$\text{(3.3)} \quad \frac{T_{n,W}}{n} = \int_{\mathbb{R}^d} |D_n(\mathbf{t})|^2 \, W(\mathbf{t}) d\mathbf{t},$$

and notice that clearly

$$\text{(3.4)} \quad |D_n(\mathbf{t})|^2 \leq \left(2 + \frac{1}{2}\|\mathbf{t}\|^2\right)^2.$$

Also from the uniform consistency of the empirical CF (see Csörgő, 1981, and Ushakov, 1999, §3.2, and p. 244) we have that

$$\text{(3.5)} \quad \varphi_n(\mathbf{t}) = \underbrace{e^{-i\mathbf{t}'\mathbf{S}_n^{-1/2}\overline{\mathbf{X}}_n}}_{(1)} \; \underbrace{\frac{1}{n}\sum_{j=1}^{n} e^{i\mathbf{t}'\mathbf{S}_n^{-1/2}\mathbf{X}_j}}_{(2)} \to \underbrace{e^{-i\mathbf{t}'\mathbf{C}^{-1/2}\boldsymbol{\mu}}}_{(1)} \; \underbrace{\phi\left(\mathbf{C}^{-1/2}\mathbf{t}\right)}_{(2)},$$

almost surely as $n \to \infty$. Now (3.2) follows by (3.3), (3.4) and (3.5), with an application of Lebesque's theorem of dominated convergence.

By changing variables in the integral figuring in the right–hand side of (3.2) one obtains

$$\Delta_W = \det(\mathbf{C}^{1/2}) \int_{\mathbb{R}^d} \left| \phi(\mathbf{u}) \left(1 + \frac{1}{2}\mathbf{u}'\mathbf{C}\mathbf{u}\right) - e^{i\mathbf{u}'\boldsymbol{\mu}} \right|^2 W(\mathbf{C}^{1/2}\mathbf{u}) d\mathbf{u}.$$

It is clear that the last integral is positive unless H_0 holds with $(\boldsymbol{\delta}, \boldsymbol{\Sigma})$ replaced by $(\boldsymbol{\mu}, \mathbf{C})$, which implies that the test which rejects the null hypothesis H_0 for large values of the test statistic $T_{n,W}$ is consistent against each alternative satisfying the assumptions of Theorem 3.1.

4 Connection with moment–type statistics

Consider the test statistic in (2.4) where the parameter–estimates are given by eqn. (3.1). By use of the expansion $e^x = 1 + x + (x^2/2!) + (x^3/3!) + ...$ in the exponential terms of (2.4) we obtain

$$\left(\frac{a}{\pi}\right)^{d/2} T_{n,a} = \frac{1}{4a} A_1 + \frac{1}{4^2 a^2} A_2 + \frac{1}{4^3 a^3} A_3 + o\left(\frac{1}{a^3}\right), \ a \to \infty, \tag{4.1}$$

where $A_m, \ m = 1, 2, 3$, are obtained after some straightforward algebra as

$$\begin{aligned} A_1 &= 2\sum_{j=1}^{n} \|\mathbf{Z}_j\|^2 - \frac{1}{n}\sum_{j,k=1}^{n} \|\mathbf{Z}_{jk}\|^2, \\ A_2 &= \frac{1}{2n}\sum_{j,k=1}^{n} \|\mathbf{Z}_{jk}\|^4 - \sum_{j=1}^{n} \|\mathbf{Z}_j\|^4 - (d+2)\left(\frac{2}{n}\sum_{j,k=1}^{n} \|\mathbf{Z}_{jk}\|^2 - nd - 2\sum_{j=1}^{n} \|\mathbf{Z}_j\|^2\right), \\ A_3 &= \frac{1}{3}\sum_{j=1}^{n} \|\mathbf{Z}_j\|^6 - \frac{1}{6n}\sum_{j,k=1}^{n} \|\mathbf{Z}_{jk}\|^6 \\ &+ (d+4)\left(\frac{1}{n}\sum_{j,k=1}^{n} \|\mathbf{Z}_{jk}\|^4 - \frac{(d+2)}{n}\sum_{j,k=1}^{n} \|\mathbf{Z}_{jk}\|^2 - \sum_{j=1}^{n} \|\mathbf{Z}_j\|^4\right). \end{aligned}$$

Now notice that due to standardization we have (see also Henze, 1997)

$$\sum_{j,k=1}^{n} \mathbf{Z}_j' \mathbf{Z}_k = 0, \qquad \sum_{j,k=1}^{n} \left(\mathbf{Z}_j' \mathbf{Z}_k\right)^2 = n^2 d$$

and

$$\sum_{j,k=1}^{n} \|\mathbf{Z}_{jk}\|^2 = 2n^2 d, \qquad \sum_{j=1}^{n} \|\mathbf{Z}_j\|^2 = nd.$$

These relations and some tedious but straightforward algebra yield $A_m = 0, \ m = 1, 2$, and

$$A_3 = \frac{1}{3}\sum_{j=1}^{n} \|\mathbf{Z}_j\|^6 - \frac{1}{6n}\sum_{j,k=1}^{n} \|\mathbf{Z}_{jk}\|^6 + (d+4)\sum_{j=1}^{n} \|\mathbf{Z}_j\|^4. \tag{4.2}$$

Compute now

$$\frac{1}{n}\sum_{j,k=1}^{n}\|\mathbf{Z}_{jk}\|^6 = -8nb_{1,d} - 12n\widetilde{b}_{1,d} + 6(d+4)\sum_{j=1}^{n}\|\mathbf{Z}_j\|^4 + 2\sum_{j=1}^{n}\|\mathbf{Z}_j\|^6,$$

where

$$\begin{aligned} b_{1,d} &= \frac{1}{n^2}\sum_{j,k=1}^{n}\left(\mathbf{Z}_j'\mathbf{Z}_k\right)^3 \\ \widetilde{b}_{1,d} &= \frac{1}{n^2}\sum_{j,k=1}^{n}\mathbf{Z}_j'\mathbf{Z}_k\|\mathbf{Z}_j\|^2\|\mathbf{Z}_k\|^2, \end{aligned}$$

and substitute this expression in (4.2) to finally obtain from (4.1)

$$\lim_{a\to\infty} a^{(d/2)+3}\, T_{n,a} = \frac{\pi^{d/2}}{8}n\left(\frac{1}{6}b_{1,d} + \frac{1}{4}\widetilde{b}_{1,d}\right) := T_{n,\infty}. \tag{4.3}$$

There is an interesting connection of the limit value in eqn. (4.3) with standard measures of multivariate skewness and kurtosis. In particular $T_{n,\infty}$ contains the measures of skewness $b_{1,d}$ introduced by Mardia (1970). Moreover $T_{n,\infty}$ coincides, apart from scaling and normalization, with the statistic obtained by Henze (1997) in the context of testing multivariate normality. Hence this limit value may be considered as a test statistic of its own and in fact, under general conditions, $T_{n,\infty}$ attains a standard asymptotic distribution. Specifically by using Theorem 2.2 of Henze (1997) it follows that under the null hypothesis of a MLD, and as $n \to \infty$,

$$n\left(\frac{1}{6}b_{1,d} + \frac{1}{4}\widetilde{b}_{1,d}\right) \to \gamma_1 Y_1 + \gamma_2 Y_2, \tag{4.4}$$

in distribution, where Y_1 and Y_2 are independent chi–squared distributed r.v.'s, with degrees of freedom d and $d(d-1)(d+4)/6$, respectively. The coefficients $\gamma_m,\ m = 1, 2$, figuring in (4.4) can be computed from the moments of the MLD as

$$\gamma_1 = \frac{3(d+4)[11d(d-2)+32]}{4(d+2)}, \quad \gamma_2 = \frac{18(2d^2-3d+6)}{(d+2)(d+4)}. \tag{4.5}$$

From (4.4) and (4.5), it follows that an asymptotic test statistic for the null hypothesis H_0 can be based on $T_{n,\infty}$. However, as it has already been pointed out, this statistic unlike $T_{n,a},\ a > 0$, is not universally consistent, and in particular it should have low power against multivariate normal alternatives, and more generally against any non–Laplacian spherically symmetric distribution.

5 Simulations

In this section we study the finite–sample performance of the test $T_{n,a}$ in eqn. (2.4), implemented via with moment estimates in eqn. (3.1). For simplicity we shall refer to this test statistic as T_a. The limit statistic $T_{n,\infty}$ (T_∞ for simplicity) will also be simulated. In particular, and since we found that the $T_{n,\infty}$ test when implemented as an asymptotic test via (4.4)–(4.5) does not respect the nominal level of significance to a satisfactory degree, we decided to also run this test as a Monte Carlo test, with both an upper tail as well as a two–tail rejection region. In fact it should be mentioned that a two–tailed test has better performance than the corresponding one–sided tests.

The simulation study is based on 10,000 samples of size $n = 50$ and $n = 100$ from several bivariate distributions ($d = 2$). These distributions are: The standard Laplace, the standard normal (N), Student's t–distribution with varying degrees of freedom, the skew–normal distribution, and a mixture of the standard Laplace with the standard normal distribution, in varying proportions. For the standard normal which is a building block for simulating random numbers following the other distributions we employed the IMSL routine DRNMVN. Then, given that the vector $\mathbf{X}$ follows a standard normal distribution, deviates $\mathbf{Y}$ for the other distributions are simulated as

- Standard Laplace: $\mathbf{Y}_L = \sqrt{w}\mathbf{X}$.
- Standard Skew Normal: $\mathbf{Y}_{SN} = \lambda|z|\mathbf{1_d} + \sqrt{1-\lambda^2}\mathbf{X}$ denoted as SN(λ).
- Student's t_m : $\mathbf{Y}_m = \left(\frac{\sqrt{S}}{m}\right)^{-1}\mathbf{X}$ denoted as t(m).
- Laplace - Normal mixture: $\mathbf{Y}_{LN} = p\mathbf{Y}_L + (1-p)\mathbf{X}$ denoted as LN(p).

with w following a standard exponential distribution, z following a standard normal distribution, S following a χ^2_m distribution and $p \in (0, 1)$.

For comparison purposes, we have also included the multivariate extension of the Kolmogorov–Smirnov test suggested by Justel et al. (1997). Given a sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$, with distribution function $F(\mathbf{x})$, the Kolmogorov–Smirnov (KS) statistic for the hypothesis $H_0 :\ F = F_0$ against $H_1 :\ F \neq F_0$ is given by

$$(5.1) \qquad D_n = \sup_{\mathbf{x}\in\mathbb{R}^d} |F_n(\mathbf{x}) - F_0(\mathbf{x})|,$$

where $F_n(\mathbf{x})$ is the corresponding EDF. The main drawbacks of KS in the multivariate setup are that D_n is computationally intensive and that it is not distribution free. On the other hand Rosenblatt (1952) proposed the following transformation $\mathbf{U} = T_0(\mathbf{X})$ of a random vector $\mathbf{X} = (X_1, X_2, ..., X_d)'$:

$$
\begin{aligned}
U_1 &= F_{01}(X_1) \\
U_i &= F_{0i}(X_i | X_1, \ldots, X_{i-1}), \quad i = 2, \ldots, d, \qquad (5.2)
\end{aligned}
$$

where $F_{01}(\cdot)$ and $F_{0i}(\cdot|\cdot)$, $i = 2, ..., d$, denote the marginal and the conditionals corresponding to F_0, respectively. It may be shown that, under the null hypothesis H_0, the random vector $\mathbf{U} := (U_1, ..., U_d)'$ is uniformly distributed on $(0,1)^d$ and, therefore, a test of H_0 may be based on the statistic:

$$(5.3) \quad d_n = \sup_{\mathbf{u} \in (0,1)^d} |G_n(\mathbf{u}) - u_1 \cdots u_d|,$$

where $G_n(\mathbf{u})$ is the EDF of the transformed sample $\mathbf{U}_1, \ldots, \mathbf{U}_n$. Justel et al. (1997) further extend this approach and propose to consider all possible permutations of the coordinates and to define the multivariate Kolmogorov–Smirnov statistic as

$$(5.4) \quad D_n^{KS} = \max_j d_n^j,$$

where d_n^j is the statistic (5.3) in which the transformed variables $\mathbf{U}^j$ are defined by (5.2) with the ordering of the variables X_i, $i = 1, \ldots, d$, permuted according to the j-th permutation, i.e.,

$$(5.5) \quad d_n^j = \sup_{\mathbf{u} \in (0,1)^d} |G_n^j(\mathbf{u}) - u_1 \cdots u_d| \quad \text{for} \quad j = 1, \ldots, d!,$$

where $G_n^j(\mathbf{u})$ is the empirical distribution function of the transformed sample $\mathbf{U}_1^j, \ldots, \mathbf{U}_n^j$.

In the following simulation study, we will use the approximate Kolmogorov–Smirnov statistic:

$$(5.6) \quad \tilde{D}_n^{KS} = \max_j \tilde{d}_n^j,$$

where the only difference to the Kolmogorov–Smirnov statistic (5.4) is that the supremum in (5.5) is replaced by the supremum calculated over the points of the transformed sample $\{\mathbf{U}_1^j, \ldots, \mathbf{U}_n^j\}$, i.e.,

$$(5.7) \quad \tilde{d}_n^j = \sup_{\mathbf{u} \in \{\mathbf{U}_1^j, \ldots, \mathbf{U}_n^j\}} |G_n^j(\mathbf{u}) - u_1 \cdots u_d| \quad \text{for} \quad j = 1, \ldots, d!.$$

Justel et al. (1997) show in a simulation study that the difference between the power of the exact and the approximate Kolmogorov–Smirnov test is negligible for $n \geq 50$.

For 5% and 10% level of significance and sample size $n = 50$ and $n = 100$, Table 1 shows the critical values of the T_a–test with several values of a. Analogous results for T_∞, and the KS statistics are also shown. Table 2 ($n = 50$), and Table 3 ($n = 100$) show the corresponding percentage of rejection (rounded to the nearest integer) for all tests and the simulated bivariate distributions. It is clear from the power results that the T_∞ test is the least powerful, and even in cases that its rejection rate exceeds that of the KS test, its use should be avoided in favor of the T_a test. In this connection the proposed test T_a is seen to be more powerful than the KS test, its superiority holding uniformly against all alternatives and for each sample size considered.

6 Extension to asymmetric Laplace and application

The asymmetric MLD contains an extra parameter $\boldsymbol{\mu} \in \mathbb{R}^d$ for skewness. The CF of this distribution is given by

$$(6.1) \quad \widetilde{\varphi}(\mathbf{t}) = \frac{e^{i\mathbf{t}'\boldsymbol{\delta}}}{1 - i\mathbf{t}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}}.$$

In view of (6.1) and by following analogous steps and reasoning to that of Section 1 we only consider the standard case with $\boldsymbol{\delta} = \mathbf{0}_d$ and $\boldsymbol{\Sigma} = \mathbf{I}_d$, and suggest the test statistic

$$(6.2) \quad \widetilde{T}_{n,a} = n \int_{\mathbb{R}^d} \left| \varphi_n(\mathbf{t}) \left(1 - i\mathbf{t}'\widehat{\boldsymbol{\mu}}_n + \frac{1}{2}\mathbf{t}'\mathbf{t} \right) - 1 \right|^2 e^{-a\|\mathbf{t}\|^2} d\mathbf{t},$$

where $\varphi_n(\mathbf{t}) = n^{-1} \sum_{j=1}^n e^{i\mathbf{t}'\mathbf{Z}_j}$.

It is easy to see that

$$\widetilde{T}_{n,a} = T_{n,a} + n \int_{\mathbb{R}^d} (\mathbf{t}'\widehat{\boldsymbol{\mu}}_n)^2 |\varphi_n(\mathbf{t})|^2 e^{-a\|\mathbf{t}\|^2} d\mathbf{t} - 2n \int_{\mathbb{R}^d} (\mathbf{t}'\widehat{\boldsymbol{\mu}}_n) S_n(\mathbf{t}) e^{-a\|\mathbf{t}\|^2} d\mathbf{t}.$$

Thus extra computation for $\widetilde{T}_{n,a}$ boils down to the calculation of the integrals

$$C_a := C_a(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_{\mathbb{R}^d} (\mathbf{t}'\boldsymbol{\alpha})^2 \cos(\mathbf{t}'\boldsymbol{\beta}) e^{-a\|\mathbf{t}\|^2} d\mathbf{t}$$

and

$$S_a := S_a(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_{\mathbb{R}^d} (\mathbf{t}'\boldsymbol{\alpha}) \sin(\mathbf{t}'\boldsymbol{\beta}) e^{-a\|\mathbf{t}\|^2} d\mathbf{t},$$

for arbitrary $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^d$. In order to compute C_a and S_a consider the CF of the multivariate normal distribution with mean $\gamma \mathbf{b}, \mathbf{b} \in \mathbb{R}^d, \gamma \in \mathbb{R}$, and covariance matrix $\mathbf{I}_d/(2a)$. From the definition of this CF we deduce the following integrals

$$C_{a,\gamma} := \int_{\mathbb{R}^d} \cos(\mathbf{t}'\mathbf{x}) e^{-a(\mathbf{x}'\mathbf{x}-2\gamma\mathbf{x}'\mathbf{b})} d\mathbf{x} = \left(\frac{\pi}{a}\right)^{d/2} \cos(\gamma \mathbf{t}'\mathbf{b}) e^{a\gamma^2 \mathbf{b}'\mathbf{b} - \frac{1}{4a}\mathbf{t}'\mathbf{t}},$$

$$S_{a,\gamma} := \int_{\mathbb{R}^d} \sin(\mathbf{t}'\mathbf{x}) e^{-a(\mathbf{x}'\mathbf{x}-2\gamma\mathbf{x}'\mathbf{b})} d\mathbf{x} = \left(\frac{\pi}{a}\right)^{d/2} \sin(\gamma \mathbf{t}'\mathbf{b}) e^{a\gamma^2 \mathbf{b}'\mathbf{b} - \frac{1}{4a}\mathbf{t}'\mathbf{t}}.$$

Differentiating twice $C_{a,\gamma}$ w.r.t. γ and setting $\gamma = 0$ yields, after some relabeling of variables,

$$C_a = \left(\frac{\pi}{a}\right)^{d/2} \frac{2a\|\boldsymbol{\alpha}\|^2 - (\boldsymbol{\beta}'\boldsymbol{\alpha})^2}{(2a)^2} e^{-\frac{1}{4a}\|\boldsymbol{\beta}\|^2}.$$

Likewise, differentiating once $S_{a,\gamma}$ w.r.t. γ and setting $\gamma = 0$ yields

$$S_a = \left(\frac{\pi}{a}\right)^{d/2} \frac{(\boldsymbol{\beta}'\boldsymbol{\alpha})}{2a} e^{-\frac{1}{4a}\|\boldsymbol{\beta}\|^2}.$$

Consequently the test statistic in eqn. (6.2) may be written as

$$\widetilde{T}_{n,a} = T_{n,a} + \frac{1}{n} \sum_{j,k=1}^{n} C_a(\widehat{\boldsymbol{\mu}}_n, \mathbf{Z}_{jk}) - 2 \sum_{j=1}^{n} S_a(\widehat{\boldsymbol{\mu}}_n, \mathbf{Z}_j)$$

We now apply the test statistic for the multivariate asymmetric Laplace distribution to real data on daily currency exchange rates. For the period from January 1, 2008 to December 31, 2008, we use a bivariate data set on two pairs of exchange rates, namely the Euro vs. the U.S. Dollar and the Japanese Yen vs. the U.S. Dollar. This data–set contains 253 observations which can be obtained from the official site of the Bank of England at www.bankofengland.co.uk. The variable of interest is the daily return, computed as $\log\left(\frac{P_t}{P_{t-1}}\right)$, for consecutive days $t-1$ and t, where P_t denotes the exchange rate at time t.

The classsical method of moments was employed for estimation of parameters. For the asymmetric MLD, this method was specified and further studied by Visk (2009). In particular, for the exchange rate data the estimates of the parameters are (left entry (resp. right entry) for Euro–U.S.D (resp. Yen–U.S.D.)),

$$\begin{aligned} \widehat{\boldsymbol{\delta}}_n &= (-0.000358, 0.001274), \\ \widehat{\boldsymbol{\mu}}_n &= (0.000586, -0.002033), \\ \widehat{\boldsymbol{\Sigma}}_n &= \begin{pmatrix} 0.0000792 & 0.0000017 \\ 0.0000017 & 0.0000865 \end{pmatrix}. \end{aligned}$$

However, and unlike the case of testing for symmetric Laplace distribution, now the null distribution of the test statistic depends on the parameter values. In particular this distribution involves the extra parameter $\boldsymbol{\mu}$, which however in the context of composite goodness–of–fit testing is considered unknown. Therefore we employ the following parametric bootstrap procedure in order to actually perform the test: (i) Conditionally on the observations compute the moment estimates and the corresponding value of the test statistic, (ii) simulate a sample from a standard asymmetric MLD by taking into account the skewness parameter obtained as estimate in the previous step, (iii) obtain the moment estimates of the asymmetric MLD from the sample values observed in the previous step, (iv) compute the value of the test statistic with the observations of the second step and the estimates obtained in step (iii). Then, the bootstrap distribution of the test statistic is produced by repeating steps (ii)–(iv) a number of times, and from the quantiles of this bootstrap distribution we decide whether the value of the test statistic obtained in step (i) is significant or not.

In the Table 4 we give the value of the test statistic for the exchange rate data. Steps (ii)–(iv) of the aforementioned bootstrap procedure were repeated 100 times, and the critical values so obtained are also shown for 5% and 10% level of significance, and for several values of the weight parameter a. By comparison it appears that we can not reject the null hypothesis of an asymmetric multivariate Laplace distribution for these data.

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Table 1: Critical values for the tests based on 10000 Monte Carlo samples of size $n = 50$ and $n = 100$ at 5% and 10% level of significance.

$n = 50$	T_a								KS	T_∞	
$a =$	0.1	0.15	0.25	0.4	0.5	0.6	0.75	1.0		Left	Right
5%	3192.42	1136.53	314.20	94.71	53.00	32.36	17.51	7.61	0.177	2.66	138.74
10%	2875.01	1003.21	271.81	81.19	45.13	27.62	14.87	6.51	0.162	3.86	107.14
$n = 100$	T_a								KS	T_∞	
$a =$	0.1	0.15	0.25	0.4	0.5	0.6	0.75	1.0		Left	Right
5%	3260.53	1152.91	313.01	93.60	51.90	31.94	17.53	7.85	0.137	3.36	204.21
10%	2920.26	1007.22	269.96	80.21	44.63	27.49	14.87	6.64	0.125	4.97	151.32

Table 2: Percentage of rejection for the tests based on 10000 Monte Carlo samples of size $n = 50$ at 5% (upper entry) and 10% (lower entry) level of significance.

	T_a								KS	T_∞
$a =$	0.1	0.15	0.25	0.4	0.5	0.6	0.75	1.0		
N	29	46	62	68	68	68	64	58	17	22
	47	64	76	81	81	81	80	74	32	35
LN(0.25)	28	45	60	66	67	66	62	56	16	21
	47	63	76	80	81	80	78	73	30	34
LN(0.50)	22	32	41	43	42	40	36	30	10	12
	37	49	58	59	58	56	53	46	21	20
LN(0.75)	7	8	8	8	7	7	6	5	4	5
	14	16	16	15	14	14	13	12	10	10
SN(0.25)	29	46	62	68	68	67	64	57	17	21
	47	64	76	81	82	81	79	74	31	35
SN(0.50)	28	45	62	68	68	67	64	57	16	22
	47	64	77	81	82	80	79	74	31	35
SN(0.75)	28	45	61	67	67	66	63	57	14	18
	47	63	76	81	81	81	79	74	28	30
SN(1.00)	73	91	98	99	99	99	99	98	69	0
	87	96	99	100	100	100	100	100	87	2
t(1)	96	97	97	97	97	97	97	97	93	91
	97	97	98	98	98	98	98	98	95	93
t(2)	30	34	37	40	41	43	45	48	34	51
	37	41	45	48	50	51	53	56	43	58
t(5)	13	16	19	18	16	15	13	11	7	10
	24	29	31	29	27	26	24	21	15	16
t(10)	21	30	37	38	37	35	31	26	10	11
	35	46	54	54	53	51	47	41	21	19

Table 3: Percentage of rejection for the tests based on 10000 Monte Carlo samples of size $n = 100$ at 5% (upper entry) and 10% (lower entry) level of significance.

	T_a								KS	T_∞
$a =$	0.1	0.15	0.25	0.4	0.5	0.6	0.75	1.0		
N	74	89	96	98	98	98	98	96	52	27
	87	96	99	99	99	99	99	99	73	43
LN(0.25)	74	89	96	98	98	98	97	96	50	26
	87	96	99	99	99	99	99	98	70	41
LN(0.50)	60	75	83	84	82	80	76	67	31	13
	76	86	91	91	90	88	86	80	49	22
LN(0.75)	13	14	14	12	11	10	9	7	6	5
	23	25	24	21	19	18	17	15	12	9
SN(0.25)	74	90	96	98	98	98	98	96	51	27
	87	96	99	99	99	99	99	99	72	43
SN(0.50)	74	90	97	98	98	98	98	96	49	27
	87	96	99	99	99	99	99	99	70	43
SN(0.75)	74	88	96	98	98	98	97	96	40	19
	86	95	99	99	99	99	99	99	63	32
SN(1.00)	99	100	100	100	100	100	100	100	100	1
	100	100	100	100	100	100	100	100	100	4
t(1)	100	100	100	100	100	100	100	100	100	97
	100	100	100	100	100	100	100	100	100	98
t(2)	48	54	62	67	69	71	72	75	54	69
	55	62	69	73	75	76	78	80	62	75
t(5)	31	37	39	34	31	29	24	19	12	11
	47	53	53	48	44	41	37	31	23	18
t(10)	55	68	77	77	75	73	67	58	26	11
	71	82	87	87	85	83	79	73	44	20

Table 4: Value of the test statistic T_a and the 5% and 10% bootstrap critical values for the currency exchange rate data.

$a =$	0.1	0.15	0.25	0.4	0.5	0.6	0.75	1.0
T_a	4423.01	1695.41	469.67	134.82	74.21	45.98	26.13	13.19
5%	8057.90	3022.51	1083.31	381.43	216.75	130.57	70.69	33.17
10%	5792.19	2094.31	605.20	194.25	115.58	72.14	38.61	17.86